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ON THE ENVELOPE OF THE WALLACE LINES OF AN INSCRIBED QUADRANGLE.

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It has long been known that the Wallace lines of a triangle envelop a three cusped hypocycloid.[1] Steggall[2] has investigated the envelope of the Wallace lines of an inscribed polygon. He gives the equation of the Wallace line in the general case and some facts about regular polygons. All he does with the quadrangle is to give the condition that the envelope be a four-cusped hypocycloid. We shall examine this case in more detail.

If perpendiculars be dropped upon the sides of a triangle from a point on the circum-circle, the three feet are collinear on a line called the Wallace line of the point with regard to the triangle. If we take an inscribed quadrangle, and omit each vertex in turn, we obtain four triangles; and the feet of the perpendiculars dropped from any point of the circle upon its four Wallace lines with regard to these triangles lie on a line called the Wallace line of the point with regard to the quadrangle. The process is capable of indefinite extension.

Let A_1, A_2, A_3, A_4 be the vertices of a quadrangle inscribed in a circle of radius R, and P any point on the circle. With the origin at the center, let θ_i and θ denote the angles measured from the positive x-axis around the circumference to A_i and P respectively. Then the equation of the Wallace line of P with regard to the quadrangle is[3]

$$\begin{aligned} x \cos \tfrac{1}{2}[\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2(\theta - \pi)] + y \sin \tfrac{1}{2}[\theta_1 + \theta_2 + \theta_3 + \theta_4 - 2(\theta - \pi)] \\ = R\{- \cos \tfrac{1}{2}[\theta_1 + \theta_2 + \theta_3 + \theta_4 - 4(\theta - \pi)] \\ + 2 \sin \tfrac{1}{2}(\theta - \theta_1) \sin \tfrac{1}{2}(\theta - \theta_2) \sin \tfrac{1}{2}(\theta - \theta_3) \sin \tfrac{1}{2}(\theta - \theta_4)\}. \end{aligned} \tag{1}$$

Let the origin be moved to the point whose coördinates are

$$\tfrac{1}{4}R(\cos \theta_1 + \cos \theta_2 + \cos \theta_3 + \cos \theta_4),\ \tfrac{1}{4}R(\sin \theta_1 + \sin \theta_2 + \sin \theta_3 + \sin \theta_4)].$$

This is the point of concurrence of the three lines joining mid-points of pairs of opposite connectors of the quadrangle. Furthermore let $\theta_1 + \theta_2 + \theta_3 + \theta_4 = \pi$, which may be done without loss of generality by a proper choice of the direction of the x-axis. All this causes the equation of the Wallace line to reduce to

$$\begin{aligned} x \sin \theta + y \cos \theta = \tfrac{1}{4}R[3 \sin 2\theta - \cos \tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3 - \theta_4) \\ - \cos \tfrac{1}{2}(\theta_1 - \theta_2 + \theta_3 - \theta_4) - \cos \tfrac{1}{2}(\theta_1 - \theta_2 - \theta_3 + \theta_4)]. \end{aligned} \tag{2}$$

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[1] *Steiner*, "Gesammelte Werke," pp. 641–647, "Über eine besondere Curve dritter Classe (und vierten Grades)." Other articles have been written about it, but this is probably the first.

[2] J. E. A. Steggall, "On the envelope of the Simson line of a polygon." *Edinb. Math. Soc. Proceedings*, Vol. 14, p. 122–126 (1896).

[3] This equation is taken from Steggall, *loc. cit.*, but with a change of notation.

Now θ is the parameter in this equation, and on differentiating with regard to θ we obtain

$$x \cos \theta - y \sin \theta = \tfrac{3}{2}R \cos 2\theta. \tag{3}$$

Now (3) is a line which meets (2) at the point where (2) touches its envelope. Moreover (3) is evidently orthogonal to (2), hence (3) envelops the evolute of the envelope of (2). But we recognize[1] that (3) envelops a hypocycloid of four cusps.

THEOREM 1. *The envelope of the Wallace lines of an inscribed quadrangle is an involute of a four cusped hypocycloid whose cusps lie on a circle of radius three times that of the circum-circle.*

Next let an angle α be defined by the equation

$$\begin{aligned} -3 \sin 2\alpha = \cos \tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3 - \theta_4) \\ + \cos \tfrac{1}{2}(\theta_1 - \theta_2 + \theta_3 - \theta_4) + \cos \tfrac{1}{2}(\theta_1 - \theta_2 - \theta_3 + \theta_4). \end{aligned}$$

Equation (2) may then be written

$$x \sin \theta + y \cos \theta = \tfrac{3}{2}R \sin (\theta + \alpha) \cos (\theta - \alpha).$$

Transform to oblique axes by

$$x = x' \cos \alpha + y' \sin \alpha,$$

$$y = x' \sin \alpha + y' \cos \alpha,$$

which turns the x-axis through an angle α and the y-axis through an angle $-\alpha$. This gives

$$\frac{x'}{\tfrac{3}{2}R \cos (\theta - \alpha)} + \frac{y'}{\tfrac{3}{2}R \sin (\theta + \alpha)} = 1$$

The intercepts on the axes are read off from the denominators in this equation, and since the angle between the axes is $\pi/2 - 2\alpha$, we easily calculate the length intercepted on the line by the axes to be $\tfrac{3}{2} \cdot R \cos 2\alpha$.

THEOREM 2. *There are two Wallace lines of an inscribed quadrangle which intercept a constant segment on all the others. These two pass through the point of concurrence of the three lines joining the mid-points of pairs of opposite sides of the quadrangle.*

This gives a very neat generation of the Wallace lines by allowing a constant segment to move with its ends sliding along two oblique lines. If, however, $\alpha = \pm \pi/4$ this generation becomes illusory since the two oblique lines would coincide. But from it we can get another generation which is never illusory, as follows: If a circle rolls on the interior of another circle of twice its radius, any point on the circumference of the rolling circle traces a diameter of the fixed circle. Therefore a chord of the rolling circle will be a line of constant length moving with its end points on two oblique lines. If the chord becomes a diameter the oblique lines become perpendicular, and if the chord becomes tangent the oblique lines coincide.

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[1] See Williamson's Differential Calculus, seventh edition, New York (1889), page 347, formula (18).

THEOREM 3. *The Wallace lines of an inscribed quadrangle are the successive positions of a chord of a circle which rolls upon the interior of a circle of twice its radius.*

In trying to picture the curve imagine first a hypocycloid of four cusps, which is one extreme case corresponding to $\alpha = 0$. Then suppose two opposite arches to grow longer at the expense of the other two which are shortened, so that the cusps stand at the vertices of a rectangle. This will be an intermediate figure. To get the other extreme let the two opposite arches swell till they become tangent to each other at the center of the curve. This corresponds to $\alpha = \pm \pi/4$. If all the vertices of the quadrangle approach coincidence the figure approaches this shape, but this is not the only case giving such a shape. It should be noted that not every involute of a four-cusped hypocycloid can be the envelope of the Wallace lines of a quadrangle if we confine ourselves to real quadrangles.

We have seen that when $\alpha = 0$ the envelope is a four-cusped hypocycloid. In terms of the θ_i this condition is

$$\cos \tfrac{1}{2}(\theta_1 + \theta_2 - \theta_3 - \theta_4) + \cos \tfrac{1}{2}(\theta_1 - \theta_2 + \theta_3 - \theta_4) + \cos \tfrac{1}{2}(\theta_1 - \theta_2 - \theta_3 + \theta_4) = 0.$$

A rotation of the axes has the effect of adding a constant to each θ_i, which obviously leaves this condition unchanged. We may therefore suppose that the axes have such a direction that $\theta_4 = 0$. Then the condition may be manipulated into the form

$$3 \cot \tfrac{1}{2}\theta_1 \cot \tfrac{1}{2}\theta_2 \cot \tfrac{1}{2}\theta_3 + \cot \tfrac{1}{2}\theta_1 + \cot \tfrac{1}{2}\theta_2 + \cot \tfrac{1}{2}\theta_3 = 0. \tag{4}$$

This looks familiar and reminds us of the following fact. If B_1, B_2, B_3 are the vertices of an equilateral triangle and Q any other point in the plane, and if φ_i denotes the angle made by the line QB_i with the bisector of the interior angle of the triangle at B_i, then

$$3 \tan \varphi_1 \tan \varphi_2 \tan \varphi_3 + \tan \varphi_1 + \tan \varphi_2 + \tan \varphi_3 = 0.$$

This shows that if we take $\theta_i = \pi - 2\varphi_i$ $(i = 1, 2, 3)$, condition (4) will be satisfied, which suggests how to solve the following

Problem. Given three points on a circle, to construct a fourth point thereon so that the Wallace lines of the quadrangle formed by the four points shall envelop a four cusped hypocycloid.

Call the given points A_2, A_3, and A_4; and measure the angles from A_4 to A_3 and from A_4 to A_2 around the circumference of the circle. Lay off half the supplements of these angles from the bisectors of two of the angles of an equilateral triangle. Join the third vertex to the intersection of the two lines thus constructed. Take the angle formed by this last line and the bisector of the interior angle at the third vertex, and lay off twice its complement from A_4 around the circumference of the circle. This locates the desired point A_1 to complete the quadrangle.

Two special cases of quadrangles which yield four cusped hypocycloids deserve mention. We easily prove by condition (4)

THEOREM 4. *If an inscribed quadrangle is a square, or if three of its vertices form an equilateral triangle, then its Wallace lines envelop a four-cusped hypocycloid.*

Suppose we cause the point P to remian fixed while the quadrangle $A_1A_2A_3A_4$-revolves about the center of the circum-circle without changing its shape. This can be done if in equation (1) we let θ be constant and replace each θ_i by $\theta_i + \lambda$. Thus the θ_i will be initial values and λ the angle through which the quadrangle has rotated, λ being now the parameter. Then by methods similar to those we have used it is not difficult to prove

THEOREM 5. *If an inscribed quadrangle revolves about the center of its circumcircle, the Wallace line of a fixed point on the circumcircle envelops a curve whose evolute is in general a two-cusped epicycloid; but if the quadrangle is a rectangle the envelope is a circle, and if it is a square*[1] *the envelope is a point. The successive positions of the Wallace line can be described as the successive positions of a line rigidly attached to a circle which rolls upon the exterior of a fixed circle of equal radius.*

It seems a pity not to generalize some of these theorems to the case of an inscribed n-gon, but only theorem 4 seems easy to extend. Steggall has found the envelope in the case of a regular n-gon to be an n-cusped hypocycloid, which generalizes part of theorem 4; and the present writer has generalized the remainder of this theorem so that it reads:

THEOREM 6. *If an inscribed n-gon is regular, or if $n - 1$ of its vertices form a regular polygon, its Wallace lines envelop an n-cusped hypocycloid.*

GEOMETRIC EXPLANATION OF A CERTAIN OPTICAL PHENOMENON.[2]

BY WM. H. ROEVER.

Description of the Phenomenon.—In the parcel checking-room of the new Union Station at Kansas City, Missouri, there is a counter covered with brass plates which have, during the course of time, received numerous scratches by the baggage which is moved around upon the counter. The scratches are not very deep and they seem to be of fairly uniform distribution in both density and direction, as one might expect them to be after the cause of their formation has been in operation for some time. The baggage room is lighted by large electric lamps which are not very close together, so that an observer near the counter may regard the illumination in his immediate neighborhood as being due to a

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[1] Steggall proves that for any regular n-gon the envelope is a point, *loc. cit.*

[2] Presented to the American Mathematical Society (Southwestern Section), December 1, 1917.